Maximal subalgebras of C*-algebras associated with periodic flows

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Abstract. We find necessary and sufficient conditions for the subalgebra of analytic elements associated with a periodic C*-dynamical system to be a maximal norm-closed subalgebra. Our conditions are in terms of the Arveson spectrum of the action. We also describe equivalent properties of the system in terms of the strong Connes spectrum and the simplicity of the crossed product.

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1 Introduction

A major motivation for the study of maximal subalgebras of commutative C*-algebras stems from an attempt to extend the Stone-Weierstrass approximation theorem to the case of non-self-adjoint subalgebras. In particular, Wermer [20] has shown if $A = \mathcal{C}(\mathbf{T})$, the C*-algebra of all continuous complex valued functions on $\mathbf{T} = \{z \in \mathbf{C} |\, |z| = 1\}$, the closed subalgebra, B, generated by all polynomials in z, $B = \left\{p(z) = \sum_{k=0}^{n} a_k z^k\}|a_k \in \mathbf{C}, n \in \mathbf{N}\right\}$ is a maximal subalgebra of A. Clearly, B is the subalgebra of A consisting of all continuous functions on $\mathbf{T}$ which can be extended to the unit disk so as to be analytic in the interior.

Let now $(A, \mathbf{T}, \alpha)$ be a periodic C*-dynamical system and $A^\alpha([0,\infty))$ the subalgebra of analytic elements, i.e. the subalgebra consisting of all elements of A with non-negative Arveson spectrum. In this paper we find necessary and sufficient conditions for $A^\alpha([0,\infty))$ to be maximal among all norm-closed subalgebras of A.

The paper is organized as follows. In Section 2 we set the notations that will be used. In Section 3, we prove that every norm-closed subalgebra that contains the subalgebra of analytic elements of an one-parameter C*-dynamical system is globally invariant. In Section 4 we introduce our spectral condition (S) and prove our main result about the maximality of the subalgebra of analytic elements of a periodic C*-dynamical system. The condition (S) is stated in

terms of the Arveson spectrum of the action α and is satisfied if, in particular, the fixed point algebra of the system is a simple C*-algebra or the Arveson spectrum contains only one positive integer and the corresponding ideal of the fixed point algebra is simple (Proposition 12). In the special case when A is the Cuntz C*-algebra O_n, $n < \infty$, [6], and α is the gauge action of $\mathbf{T}$ on A, the fixed point algebra of this system is the uniformly hyperfinite C*-algebra n^∞ which is a simple C*-algebra and therefore our condition, (S), is satisfied. We then prove that Condition (S) is equivalent with a deeper property of the system $(A, \mathbf{T}, \alpha)$ which involves the strong Connes spectrum defined by Kishimoto [10] and the simplicity of the crossed products. In Theorem 13, we prove that the condition (S) is equivalent with the maximality of the subalgebra of analytic elements of the system. Finally, in Proposition 14 we describe the special case of our spectral condition (S) in which the Arveson spectrum contains only one positive integer and then give examples when this situation may occur.

In the particular case when $A = C(\mathbf{T})$, the algebra of continuous functions on $\mathbf{T} = \{z \in \mathbf{C} | \, |z| = 1\}$, and α is the action of $\mathbf{T}$ on A by translations, the result of J. Wermer that was cited above, follows immediately.

In [[17], Corollary 3.12.], Solel states a necessary and sufficient condition for the maximality of the subalgebra of analytic elements associated with a periodic W*-dynamical system $(M, \mathbf{T}, \alpha)$.This condition is similar with our condition (S) and is satisfied if, in particular, the fixed point algebra, M^α is a von Neumann factor. We mention that a von Neumann factor can be either a simple C*-algebra (finite factors) or a prime C*-algebra (infinite factors). Our results show that for a periodic C*-dynamical system, the maximality of the subalgebra of analytic elements is related only to the simplicity of some ideals of the fixed point algebra and not to their primeness.

In [18] it is discussed the maximality of analytic elements of an one parameter W*-dynamical system, $(M, \mathbf{R}, \alpha)$ with two additional conditions:

M is a σ-finite von Neumann algebra and

$\mathcal{Z}_M \cap M^\alpha = \mathbf{C}\mathbf{1}_M$ where $\mathcal{Z}_M$ is the center of M amd M^α is the fixed-point algebra of the system.

Our results and methods for periodic C*-dynamical systems and their particular cases for periodic W*-dynamical systems do not require any of the above conditions.

A related but different direction of studying the subalgebras of analytic elements, that of subdiagonal algebras, has been initiated in [1]. The study of subdiagonal algebras has been developed further in [9], for both W* and C*-dynamical systems and in [8], [4], [15] among others, for W*-dynamical systems. Obviously, our condition (S) implies that $A^\alpha([0, \infty))$ satisfies the less stringent condition of maximality among subdiagonal algebras.

2 Notations and preliminary results: Spectral subspaces for one-parmeter dynamical systems

Let $(X, \mathcal{F})$ be a dual pair of Banach spaces ([2], [12], [21], [22]). As in [21], denote by $B_{\mathcal{F}}(X)$ the linear space of all $\mathcal{F}-$ continuous linear operators on X. A one-parameter group $\{U_t\}_{t\in\mathbf{R}} \subset B_{\mathcal{F}}(X)$ is called $\mathcal{F}-$continuous if for each $x \in X$ and $\varphi \in \mathcal{F}$, the function $t \mapsto \langle U_t x, \varphi\rangle$ is continuous. $\{U_t\}$ is called bounded if $\sup_{t\in\mathbf{R}} \|U_t\| < \infty$. Examples of dual pairs of Banach spaces and one-parameter groups considered in this paper include:

$X = M$, a von Neumann algebra, $\mathcal{F} = M_*$ its predual and $\{U_t\} = \{\alpha_t\}$ a one-parameter group of automorphisms of M such that $t \mapsto \langle \alpha_t(x), \varphi\rangle$ is continuous for every $x \in X$ and $\varphi \in M_*$;

$X = A$, a C*-algebra, $\mathcal{F} = A^*$ its dual and $\{U_t\} = \{\alpha_t\}$ a one-parameter group of automorphisms of A such that $t \mapsto \alpha_t(x) \in A$ is continuous for every $x \in A$;

$X = H$, a Hilbert space and $\{U_t\} = \{u_t\}$ a strongly continuous one-parameter group of unitary operators on H.

An element $\gamma \in \widehat{\mathbf{R}}$ is said to be an essential point for U if for every neighborhood V of γ,there is $f \in L^1(\mathbf{R})$ such that $support(\widehat{f})$ is compact and is included in V and $U_f = \int f(t)U_t dt \neq 0$. Here $\widehat{f}$ is the Fourier transform of f. The Arveson spectrum of U is by definition, [3]

$$sp(U) = \Big\{\gamma \in \widehat{\mathbf{R}} | \gamma \text{ is an essential point for } U\Big\}$$

If $sp(U)$ is a non trivial discrete subset of $\mathbf{R}$, then $\{U_t\}$ is called periodic and can be viewed as a compact group $\{U_g\}_{g\in G}$, where $G = \mathbf{R}/sp(U)^{\perp}$, where $sp(U)^{\perp} = \big\{r \in \mathbf{R} | e^{ir\gamma} = 1, \gamma \in sp(U)\big\}$. The group G can be identified with $\mathbf{T} = \{z \in \mathbf{C} | \, |z| = 1\}$.

Let $x \in X$ and $\gamma \in \widehat{\mathbf{R}}$. Then, [3], γ is called an U-essential point for x if for every neighborhood V of γ,there is $f \in L^1(\mathbf{R})$ such that $support(\widehat{f})$ is compact and is included in V and $U_f(x) = \int f(t)U_t x dt \neq 0$. Following [3], we define the Arveson spectrum of x,

$$sp_U(x) = \Big\{\gamma \in \widehat{\mathbf{R}}) | \gamma \text{ is an } U\text{-essential point for } x\Big\}$$

If $E \subset \widehat{\mathbf{R}}$ is a closed set, define the spectral subspace, [3]

$$X^U(E) = \{x \in X | sp_U(x) \subset E\}$$

If $O \subset \widehat{\mathbf{R}}$ is an open set and $O = \cup E_\lambda$ where $\{E_\lambda\}$ is an increasing net of closed sets such that $O = \cup E_\lambda = O = \cup E_\lambda^\circ$, we will denote $X^U(O) = \overline{\cup X^U(E_\lambda)}$, where the closure is taken in the $\mathcal{F}-$topology of X, [20]. Note that the notations used in [20] are slightly different, but clearly defined. We mention that the spectral subspaces $A^\alpha(E)$ can be defined analogously for every C*-dynamical system (A, G, α) with G a locally compact abelian group [[3], [12]].

If $(A, \mathbf{R}, \alpha)$ (respectively $(A, \mathbf{T}, \alpha)$) is a C*-dynamical system, $A^\alpha([0,\infty))$ is said to be the subalgebra of analytic elements of the system.

Let us now recall some concepts from [[21], Section 4]:

We say that $\varphi \in \mathcal{F}$ is absolutely continuous relative to $\{U_t^*\}$ if $t \mapsto U_t^*\varphi$ is norm continuous. The set of all absolutely continuous elements of $\mathcal{F}$ relative to $\{U_t^*\}$ is a norm-closed linear subspace of $\mathcal{F}$ and will be denoted in this paper by $\mathcal{F}_{norm}^{U^*}$.

The following definitions are from [[21], Section 5]. Let H be a Hilbert space, $S \subset B(H)$ and $K \subset H$ be a closed linear subspace; then we denote by $[SK]$ the closed linear span of $SK = \{x\xi | x \in S, \xi \in K\}$.

Remark 1 *Let $K \subset H$ be a closed subspace and $K_\lambda = \cap_{\lambda<\mu}[M^\alpha((-\infty,\mu])K]$, $\lambda \in \mathbf{R}$. Then we have:*
i) $K_{\lambda_1} \subset K_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$
ii) $K_\lambda = \cap_{\lambda<\mu} K_\mu$
iii) $M^\alpha((-\infty,\nu])K_\lambda \subset K_{\nu+\lambda}$
and, denoting $K_{-\infty} = \cap_\lambda K_\lambda$, $K_\infty = \overline{\cup_\lambda K_\lambda}$ and e_λ the orthogonal projection onto $K_\lambda, \lambda \in [-\infty,\infty]$, we have
iv) e_∞, $e_{-\infty} \in M'$.

Proof. i) and ii) are immediate from definitions. iii) is a consequence of the fact that $M^\alpha((-\infty,\nu])M^\alpha((-\infty,\lambda]) \subset M^\alpha((-\infty,\nu+\lambda])$ [21, Theorem 1.6]. Finally, iv) follows from the easily checked fact that the two subspaces $e_\infty H$ and $e_{-\infty}H$ are invariant for every $x \in M$. ∎

A closed linear subspace $K \subset H$ is called invariant relative to $\{\alpha_t\}$ if $\cap_{\lambda>0}[M^\alpha((-\infty,\lambda])K] = K$

K is called doubly invariant if $\cap_{\lambda\in\mathbf{R}}[M^\alpha((-\infty,\lambda])K] = K$

K is called simply invariant if $\cap_{\lambda\in\mathbf{R}}[M^\alpha((-\infty,\lambda])K] = \{0\}$.

If K is invariant relative to $\{\alpha_t\}$, e_∞ is called the support of K [[21], Section 5].

3 Subalgebras containing the analitic elements, $A^\alpha([0,\infty))$

Let $(A, \mathbf{R}, \alpha)$ be a one-parameter C*-dynamical system. In this section we will prove that every norm-closed subalgebra $B \subset A$ such that $A^\alpha([0,\infty)) \subset B$ is α-invariant.

In [11] it is shown that all σ-finite von Neumann algebras are hereditarily reflexive (as stated in the following lemma). Our next result is the extension of [[11], Corollary 3.7] to the general case of not necessarily σ- finite von Neumann algebras.

Lemma 2 *Let $M \subset B(H)$ be a von Neumann algebra in standard form., $x \in M$, $S \subset M$ a w*-closed linear subspace such that:*

$$xS\xi \subset \overline{S\xi}, \forall \xi \in H$$

Then $xS \subset S$. If, in particular, $\mathbf{1}_M \in S$, then $x \in S$.

Proof. Let $\varphi \in M_*$ be such that $\varphi|_S = 0$. Then, $\exists \xi, \eta \in H$ such that $\varphi = \omega_{\xi,\eta}$, that is $\varphi(z) =< z\xi, \eta >, \forall z \in M$, where $< \cdot, \cdot >$ is the scalar product in H [[19], 5.16 and 10.25]. $\varphi|_S = 0$ implies that $\eta \perp S\xi$, hence $\eta \perp \overline{S\xi}$. Since $xS\xi \subset \overline{S\xi}$ it follows that $\eta \perp xS\xi$. Therefore, $\varphi(xy) =< xy\xi, \eta >= 0$ $\forall y \in S$. Applying the Hahn-Banach theorem, it follows that $xy \in S$, $\forall y \in S$. ∎

The next Lemma is an extension of [18, Proposition 2.1] to the more general case of not necessarily σ-finite von Neumann algebras.

Lemma 3 *Let $(M, \mathbf{R}, \alpha)$ be a W*-dynamical system. Let N be a w*-closed subalgebra of M containing $M^\alpha((-\infty, 0))$ and $\mathbf{1}_M$. Then, N is α-invariant.*

Proof. By Lemma 2, we have to prove that if $M \subset B(H)$ is in standard form, $x \in N$, $t \in \mathbf{R}$ and $K \subset H$ is a closed linear subspace with $NK \subset K$, then $\alpha_t(x)K \subset K$. Let now $K \subset H$ be a closed linear subspace such that $NK \subset K$. Then, it is clear that, with the notations in Remark 1, $K_0 = \cap_{\mu>0}[M^\alpha((-\infty, \mu])K]$ is invariant relative to $\{\alpha_t\}$. By [[21], Theorem 5.1] $K_{-\infty}$ is a doubly invariant subspace relative to $\{\alpha_t\}$ and $K_0 \ominus K_{-\infty}$ is simply invariant with support $e_\infty - e_{-\infty}$. By [[21], Theorem 5.2] $[M^\alpha((-\infty, \lambda])K_{-\infty}] = K_{-\infty}$ $\forall \lambda \in \mathbf{R}$. By [[21], Theorem 5.3] there exists a strongly continuous one-parameter group of unitaries $\{u_t\}$ on H,commuting with $e_\infty - e_{-\infty}$ such that

$$\begin{aligned} H^u((-\infty, \lambda]) &= K_\lambda \oplus (H \ominus K_\infty) \text{ and} \\ \alpha_s(y)(e_\infty - e_{-\infty}) &= u_s y u_s^*(e_\infty - e_{-\infty}), \forall y \in M, \forall s \in \mathbf{R}. \end{aligned}$$

Let e denote the orthogonal projection on K. If $\lambda < 0$, then, for $\lambda < \mu < 0$, $M^\alpha((-\infty, \mu])K \subset NK \subset K$ by our assumption on the subspace K and thus $K_\lambda \subset K$. If $\lambda \geq 0$, then, for $\lambda < \mu$, $\mathbf{1}_M K = K \subset M^\alpha((-\infty, \mu])K$, hence $K_\lambda \supset K$. Therefore e commutes with all e_λ, hence with all orthogonal projections onto spectral subspaces $H^u((-\infty, \lambda])$. It follows that e commutes with every u_s, $s \in \mathbf{R}$. Let now $\xi \in K$. Then $\xi = \xi_1 + \xi_2$, $\xi_1 \in K_{-\infty}$, $\xi_2 \in K \ominus K_{-\infty}$ and we have

$$\begin{aligned} \alpha_t(x)\xi_1 &= \alpha_t(x)e_{-\infty}\xi_1 = e_{-\infty}\alpha_t(x)\xi_1 \in K_{-\infty} \subset K \text{ and} \\ \alpha_t(x)\xi_2 &= \alpha_t(x)(e_\infty - e_{-\infty})\xi_2 = u_t x u_t^*(e_\infty - e_{-\infty})\xi_2 = \\ &= u_t x u_t^*(e - e_{-\infty})\xi_2 = u_t x u_t^* e \xi_2 - e_{-\infty} u_t x u_t^* \xi_2 = \\ &= e u_t x e u_t^* \xi_2 - e_{-\infty} u_t x u_t^* \xi_2 \in K \end{aligned}$$

Hence $\alpha_t(x)\xi \in K$. Therefore $\alpha_t(x) \in K$. ■

Next, we will prove an analog of Lemma 3 for C*-dynamical systems. Let A be a C*-algebra and $\mathcal{F}$ a Banach space in duality with A [[21], page 88]. Let α be an $\mathbf{R} - \sigma(A, \mathcal{F})-$continuous one-parameter group of $\sigma(A, \mathcal{F})-$continuous automorphisms of A.

Lemma 4 *Let $B \subset A$ be a $\sigma(A, \mathcal{F})-$closed subalgebra containing $A^{\alpha}((-\infty, 0))$. Then $\mathbf{C}\mathbf{1}_{\widetilde{A}} + B$ is $\alpha-$invariant, where $\mathbf{1}_{\widetilde{A}}$ is the unit of the multiplier algebra of A.*

Proof. By the discussion following [[21], Corollary 4.2.], we have that $M = (\mathcal{F}^{\alpha^*}_{norm})^*$ is a W*-algebra that, naturally contains A, such that the action α of $\mathbf{R}$ on A extends by w*-continuity to a w*-continuous action of $\mathbf{R}$ on M, still denoted by α. Then, by [[21],Theorem 4.1], we have:

$$\mathcal{F}^{\alpha^*}((-\infty, \lambda]) \subset \mathcal{F}^{\alpha^*}_{norm} \text{ and}$$
$$\mathcal{F}^{\alpha^*}([\lambda, \infty)) \subset \mathcal{F}^{\alpha^*}_{norm}, \forall \lambda \in \mathbf{R}$$

Put $N = \mathbf{C}\mathbf{1}_M + \overline{B}^{w*}$. Then N is a w*-closed subalgebra of M that contains $\mathbf{1}_M$. Let now $\varphi \in \mathcal{F}^{\alpha^*}_{norm}$ be such that $\varphi|_N = 0$. Then $\varphi(A^{\alpha}((-\infty, 0)) = \{0\}$ and applying [[21], Corollary 1.5], it follows that $\varphi \in \mathcal{F}^{\alpha^*}([0, \infty))$. Hence $\varphi \in \mathcal{F}^{\alpha^*}_{norm}([0, \infty))$ and using again [[21], Corollary 1.5], we get $\varphi|_{M^{\alpha}((-\infty,0))} = 0$. Hence, N contains $M^{\alpha}((-\infty, 0))$. By Lemma 3, N is α-invariant. Identifying $\widetilde{A}$ with $\mathbf{C}\mathbf{1}_M + A$,it follows that $N \cap \widetilde{A}$ is α-invariant. Since, obviously, $N \cap \widetilde{A} \supset \mathbf{C}\mathbf{1}_M + B$, in order to prove that $\mathbf{C}\mathbf{1}_M + B$ is α-invariant, it is sufficient to prove that $N \cap \widetilde{A} \subset \mathbf{C}\mathbf{1}_M + B$. By the Hahn-Banach theorem it is enough to prove that if $\psi \in \mathcal{F}$ vanishes on the $\sigma(\widetilde{A}, \mathcal{F})$-closed linear subspace $\mathbf{C}\mathbf{1}_M + B$, then it vanishes also on $N \cap \widetilde{A}$. Let $\psi \in \mathcal{F}$ be such that $\psi|_{\overline{\mathbf{C}\mathbf{1}_M + B}^{\sigma}} = 0$. Then, since $A^{\alpha}((-\infty, 0)) \subset B$, it follows that $\psi|_{A^{\alpha}((-\infty,0))} = 0$. Applying [[21], Corollary 1.5 and Theorem 4.1] we get $\psi \in \mathcal{F}^{\alpha^*}([0, \infty)) \subset \mathcal{F}^{\alpha^*}_{norm} = M_*$. Therefore, $\psi|_B = 0$ implies $\psi|_{\overline{B}^{w^*}} = 0$. It follows that $\psi|_N = 0$ and thus $\psi|_{N \cap \widetilde{A}} = 0$ and we are done. ■

Remark 5 *In the hypotheses of Lemma 4, if A is not unital, then B itself is α-invariant.*

Proof. Indeed, in this case $N \cap A$ is $\alpha-$invariant and $N \cap A = B$. ■

Corollary 6 *Let $(A, \mathbf{R}, \alpha)$ be a one parameter C*-dynamical system. If $B \subset A$ is a norm-closed subalgebra containing $A^{\alpha}([0, \infty))$, then B is $\alpha-$invariant.*

Proof. Since, for every $\lambda \geqq 0$, $A^{\alpha}((\lambda, \infty)) = A^{\alpha}((-\infty, \lambda))^*$ and $sp(\mathbf{1}) = \{0\}$ if $\mathbf{1} \in A$, the Corollary follows from Lemma 4 and Remark 5. ■

$A^{\alpha}([0, \infty))$ will be called the algebra of analytic elements of the system $(A, \mathbf{R}, \alpha)$.

4 Periodic C*-dynamical systems: The maximality of $A^{\alpha}([0,\infty))$

Let now $(C, \mathbf{G}, \alpha)$ be a C*-dynamical system with $\mathbf{G}$ compact abelian and $\widehat{\mathbf{G}}$ its dual group of. For $\gamma \in \widehat{\mathbf{G}}$, denote $C_\gamma = C^\alpha(\{\gamma\})$. Then it is well known and easy to see that $C_\gamma = \left\{ \int_{\mathbf{G}} \overline{< g, \gamma >} \alpha_g(x) dg | x \in C \right\} = \{x \in C | \alpha_g(x) =< g, \gamma > x\}$ .Here, $< g, \gamma >$ denotes the Pontryagin duality. Then, the Arveson spectrum of the action α is $sp(\alpha) = \left\{ \gamma \in \widehat{\mathbf{G}} | C_\gamma \neq \{0\} \right\}$ and C is the closed linear span of $\cup \{C_\gamma | \gamma \in sp(\alpha)\}$ As remarked above, $C_{-\gamma} = C_\gamma^*$ and thus $-\gamma \in sp(\alpha)$ if $\gamma \in sp(\alpha)$. For $\gamma = e$, the neutral element of $\widehat{\mathbf{G}}$, we will denote $C_e = C^\alpha$. It is immediate to see that for every $\gamma \in sp(\alpha)$, $C_\gamma C_{-\gamma} = C_\gamma C_\gamma^* = \left\{ \sum_{finite} c_k d_k^* | c_k, d_k \in C_\gamma \right\}$ is a two sided ideal of C^α. In particular if $\mathbf{G} = \mathbf{T}$ is the set of complex numbers of modulus 1, and $\alpha : \mathbf{T} \to Aut(C)$ an action of $\mathbf{T}$ on C, then, $\widehat{\mathbf{T}} = \mathbf{Z}$ where $\mathbf{Z}$ is the group of integers. For every $n \in \mathbf{Z}$ denote by $C_n = C^\alpha(\{n\})$. In this case the algebra of analytic elements of the system $(C, \mathbf{T}, \alpha)$, $C^\alpha([0,\infty)) \subset C$, is the closed linear span of $\cup \{C_n | n \in sp(\alpha), n \geq 0\}$.

Remark 7 *Let $(C, \mathbf{G}, \alpha)$ be a C*-dynamical system with $\mathbf{G}$ compact abelian.Then, the following statements hold:*
i) every approximate identity $\{e_\lambda\}$ of C^α is an approximate identity of C.
ii) If $\{e_\lambda\}$ is an approximate identity of the two sided ideal $\overline{C_\gamma C_{-\gamma}}$, then $\{e_\lambda\}$ is a right approximate identity of $C_{-\gamma}$.
iii) $\overline{C_\gamma C_{-\gamma}}$ has an approximate identity $\{e_\lambda\} \subset C_\gamma C_{-\gamma}$.

Proof. i) Let $\gamma \in sp(\alpha)$ and $c \in C_\gamma$Then, since $c^* c \in C^\alpha$, we have

$$\lim_\lambda \|ce_\lambda - c\|^2 = \lim_\lambda \|(ce_\lambda - c)^*(ce_\lambda - c)\| = \lim_\lambda \|e_\lambda c^* c e_\lambda - e_\lambda c^* c - c^* c e_\lambda + c^* c\| = 0$$

Since C is the closed linear span of $\cup \{C_\gamma | \gamma \in sp(\alpha)\}$ we are done.

The proof of the statement ii) is identical. Finally, since $C_\gamma C_{-\gamma}$ is a dense two sided ideal of $\overline{C_\gamma C_{-\gamma}}$, iii) follows from [[7], Proposition 1.7.2.]. ■

We will denote by $\mathcal{H}^\alpha(C)$ the set of all non zero, $\alpha-$invariant hereditary subalgebras of C. Let $\widetilde{\Gamma}(\alpha)$ be the strong Connes spectrum of Kishimoto, [10], namely,

$$\widetilde{\Gamma}(\alpha) = \left\{ n \in \mathbf{Z} | \overline{D_n D_n^*} = D^\alpha, \forall D \in \mathcal{H}^\alpha(C) \right\}$$

Then, $\widetilde{\Gamma}(\alpha)$ is a semi group and it plays an important role in checking the simplicity of the C*-crossed product [10]. The role of $\widetilde{\Gamma}(\alpha)$ for C*-dynamical systems is similar in many situations to the one of the Connes spectrum, $\Gamma(\alpha)$ for W*-dynamical systems.

If $E \subset \widehat{\mathbf{G}}$, denote $E_\perp = \{g \in \mathbf{G} | < g, \gamma >= 1, \forall \gamma \in E\}$. If $F \subset G$, denote $F^\perp = \left\{ \gamma \in \widehat{\mathbf{G}} | < g, \gamma >= 1, \forall g \in F \right\}$.

Lemma 8 *Let $(C, \mathbf{G}, \alpha)$ be a C*-dynamical system with $\mathbf{G}$ compact abelian. Then the following are equivalent:*
i) C^α is simple and $sp(\alpha)$ is a subgroup of $\widehat{\mathbf{G}}$
ii) $sp(\alpha)$ is a subgroup of $\widehat{\mathbf{G}}$ and the crossed product $C \rtimes_{\alpha^\bullet} \mathbf{G}/sp(\alpha)_\perp$ is a simple C-algebra*
iii) C is α-simple and $sp(\alpha) = \widetilde{\Gamma}(\alpha)$

Proof. Since all three conditions imply that $sp(\alpha)$ is a group, by Pontryagin duality it follows that $(sp(\alpha)_\perp)^\perp = sp(\alpha)$ in all three conditions. Replacing the system with $(C, \mathbf{G}/sp(\alpha)_\perp, \alpha^\bullet)$, if necessary, we may assume that α is faithful and $sp(\alpha) = \widehat{\mathbf{G}}$. Here, $\alpha^\bullet$ denotes then quotient action of $\mathbf{G}/sp(\alpha)_\perp$ on C. By [[14], Corollary 3.8.], condition i) with $sp(\alpha) = \widehat{\mathbf{G}}$ is equivalent with ii). By [[10] Thm. 3.5.], ii) is equivalent with iii). ■

Let $(A, \mathbf{T}, \alpha)$ be a C*-dynamical system. In the rest of this paper we will assume that the action α is non trivial, that is, $sp(\alpha) \neq \{0\}$. Consider the following spectral property of the system $(A, \mathbf{T}, \alpha)$:

(S) There exists a non zero, closed, two sided ideal $J \subset A^\alpha$ which is a simple C*-algebra and such that for every $n \in sp(\alpha)$, $n \geq 1$ $\overline{A_n A_{-n}} = J$.

It is obvious that if the fixed point algebra A^α is simple, then the condition (S) is satisfied. In particular, if $A = O_n$, $n < \infty$ is the Cuntz algebra generated by the isometries $\{S_i | i = 1, 2, ...n\}$ [6] and α is the gauge action $\alpha_z(S_i) = zS_i$, $1 \leq i \leq n$, $z \in \mathbf{T}$, the fixed point algebra, O^α is simple and therefore the condition (S) is satisfied.

We will state and prove conditions that are equivalent to (S) and show that these equivalent conditions are necessary and sufficient for the algebra of analytic elements, $A^\alpha([0, \infty))$, to be a maximal subalgebra of A.

Lemma 9 *Assume that condition (S) is satisfied. Then, if $n, k \in sp(\alpha)$, $n, k \geq 1$, then $n - k \in sp(\alpha)$.*

Proof. From the definition of spectral subspaces, we immediately infer that $A_{-k}A_n \subset A_{n-k}$. If we show that $A_{-k}A_n \neq \{0\}$, the conclusion of the lemma follows. Assume to the contrary that $A_{-k}A_n = \{0\}$. Multiplying the previous equality to the left by A_k and to the right by A_n we get $A_k A_{-k} A_n A_{-n} = \{0\}$. Hence $JJ = \{0\}$, contradiction since $JJ = J^2 = J \neq \{0\}$ by condition (S). ■

The following Proposition, describes the Arveson spectrum, $sp(\alpha)$, under condition (S).

Proposition 10 *Assume that condition (S) is satisfied. Then either there exists an $n \in \mathbf{Z}$, $n > 0$ such that $sp(\alpha) = \{-n, 0, n\}$ or $sp(\alpha)$ is a subgroup of $\mathbf{Z}$.*

Proof. Assume that the first alternative in the statement of the Proposition does not hold. As $sp(\alpha) \neq \{0\}$, let n_0 be the smallest positive element of $sp(\alpha)$. By assumption, there is an $n > n_0$, $n \in sp(\alpha)$. Let n_1 be the smallest such n.

By Lemma 9, it follows that $n_1 - n_0 \in sp(\alpha)$. Since n_0 is the smallest positive element of $sp(\alpha)$ we have $n_1 \geq 2n_0$. Let k be the positive integer such that $k \geq 2$ and $kn_0 \leq n_1 < (k+1)n_0$. Since n_0 is the least positive element of $sp(\alpha)$, applying Lemma 9, it follows that $n_1 = kn_0$. Another application of Lemma 9 gives $n_1 - n_0 = (k-1)n_0 \in sp(\alpha)$. Since $n_1 = kn_0$ is the smallest element in $sp(\alpha)$ such that $n_1 > n_0$ and $(k-1)n_0 \in sp(\alpha)$, it follows that that $k = 2$ and thus $n_1 = 2n_0$. Condition (S) implies that $\overline{A_{2n_0}A_{-2n_0}} = \overline{A_{n_0}A_{-n_0}} = J$. By multiplying the previous double eqality by A_{-n_0} to the left and by A_{n_0} to the right and using the fact that $A_nA_k \subset A_{n+k}$ for every $n, k \in sp(\alpha)$, it immediately follows that $\overline{A_{-n_0}A_{n_0}} \subset J$ and since J is a simple C*-algebra, $\overline{A_{-n_0}A_{n_0}} = J$. By Remark 7 ii), any approximate identity $\{e_\lambda\}$ of J is a right approximate identity of A_{n_0}. Hence $A_{n_0} = A_{n_0}J$ and $\overline{A_{n_0}JA_{-n_0}} = J$. Thus $A_{n_0}\overline{A_{2n_0}A_{-2n_0}}A_{-n_0} = J \neq \{0\}$ and so $A_{n_0}A_{2n_0} \neq \{0\}$. Therefore $3n_0 \in sp(\alpha)$. By induction it follows that $sp(\alpha) = \{kn_0 | k \in \mathbf{Z}\}$ and we are done. ■

Remark 11 *Assume that condition (S) holds and $sp(\alpha)$ is a subgroup of $\mathbf{Z}$. Then, $\overline{A_nA_{-n}} = J$ for every $n \in sp(\alpha)$, $n \neq 0$ (not only for $n > 0$).*

Proof. Follows from the proof of Proposition 10. ■

The following result gives equivalent formulations for the condition (S).

Proposition 12 *Let $(A, \mathbf{T}, \alpha)$ be a C*-dynamical system with $sp(\alpha) \neq (0)$. The following conditions are equivalent:*
i) The condition (S) holds
ii) Either
ii1) There is a positive integer n_0 such that $sp(\alpha) = \{-n_0, 0, n_0\}$ and $J_{n_0} = \overline{A_{n_0}A_{-n_0}}$ is a simple C-subalgebra of A^α*
or
ii2) There exists an α-invariant hereditary C-subalgebra, $C \in \mathcal{H}^\alpha(A)$, such that the following conditions hold:*
a) $A_n = C_n$ for every $n \in sp(\alpha)$,$n \neq 0$, hence $sp(\alpha) = sp(\alpha|_C)$
b) C is α-simple
c) $sp(\alpha|_C) = \widetilde{\Gamma}(\alpha|_C)$.

Proof. i) $\Rightarrow$ ii) Assume that the condition (S) holds and ii1) is not satisfied. Then, Proposition 10 implies that $sp(\alpha)$ is a subgroup of $\mathbf{Z}$. By Remark 11, $\overline{A_nA_{-n}} = J$ for every $n \in sp(\alpha)$, $n \neq 0$. Let $C = \overline{JAJ}$. Then, clearly, $C \in \mathcal{H}^\alpha(A)$ and $C_n = A_n$ for every $n \in sp(\alpha)$,$n \neq 0$, so i) $\Rightarrow$ ii2), a). Since $C^\alpha = J$ is a simple C*-algebra, it follows that C is α-simple. Hence i) $\Rightarrow$ ii2), b). Since $C^\alpha = J$ is simple, and $sp(\alpha|_C)$ is a subgroup, applying Lemma 8, it follows that i) $\Rightarrow$ ii2) c) and hence the implication i) $\Rightarrow$ ii) is proven.

ii) $\Rightarrow$ i) Trivially, ii1)$\Rightarrow$i). Assume now that ii2) holds. Then, since $sp(\alpha)$ is closed to taking opposites and $\widetilde{\Gamma}(\alpha)$ is a semigroup [[10], Proposition 2.1.], it follows that $sp(\alpha|_C)$ is a group. By Lemma 8, ii2) b) and ii2) c) imply that C^α is simple and $sp(\alpha)$ is a subgroup of $\mathbf{Z}$. Therefore for every $n \in sp(\alpha)$,$n \neq 0$, we have $\overline{C_nC_{-n}} = C^\alpha$. Since $A_n = C_n$ for every such n, i) follows if we set $J = C^\alpha \subset A^\alpha$. ■

We can now state our main result:

Theorem 13 *Let $(A, \mathbf{T}, \alpha)$ be a C*-dynamical system with $\alpha \neq id$. Then the following conditions are equivalent:*
i) The condition (S) is satisfied
ii) $A^\alpha([0,\infty))$ is a maximal norm-closed subalgebra of A

Proof. i) $\Rightarrow$ ii) Let $B \subseteq A$ be a norm closed subalgebra such that $A^\alpha([0,\infty)) \subsetneqq B$. By Corollary 6, B is α-invariant. Since $B \neq A^\alpha([0,\infty))$, there exists $n \in \mathbf{N}$ such that $B_{-n} \neq \{0\}$. Since $A^\alpha([0,\infty)) \subset B$ we have $B_n = A_n$ and $J \subset A^\alpha \subset B$. It follows that $B_n B_{-n}$ is a non zero two sided ideal of J. Since by i), J is simple, it follows that the ideal $B_n B_{-n}$ is dense in J. Therefore, by Remark 7 iii), there exists an approximate identity $\{e_\lambda\}$ of J, $\{e_\lambda\} \subset B_n B_{-n}$. From Remark 7 ii), it follows that $\{e_\lambda\}$ is a right approximate identity of A_{-n}. Let now $a \in A_{-n}$ be arbitrary. Since $A_{-n} B_n \subset A^\alpha \subset B$, we have that $ae_\lambda \in B$ for every λ. Since $\{e_\lambda\}$ is a right approximate identity of A_{-n}, it follows that $a = \lim ae_\lambda \in B$. Hence $B_{-n} = A_{-n}$. According to Proposition 10, either there is an $n \in \mathbf{N}$ such that $sp(\alpha) = \{-n, 0, n\}$ or $sp(\alpha)$ is a subgroup of $\mathbf{Z}$. If $sp(\alpha) = \{-n, 0, n\}$, then the above discussion implies that $B = A$. Assume now that $sp(\alpha)$ is a non zero subgroup of $\mathbf{Z}$, so there is $n_0 \in \mathbf{N}$ such that $sp(\alpha) = \{kn_0 | k \in \mathbf{Z}\}$. If $A^\alpha([0,\infty)) \subsetneqq B \subseteqq A$ and $n \in \mathbf{N}$ is such that $B_{-n} \neq \{0\}$, then $n \in \{kn_0 | k \in \mathbf{Z}\}$. Let $k_0 \in \mathbf{N}$ be the smallest natural number such that $B_{-k_0 n_0} \neq \{0\}$. We claim that $k_0 = 1$. Assume that $k_0 > 1$. Then, $B_{-k_0 n_0} A_{n_0} \neq \{0\}$ since, otherwise $B_{-k_0 n_0} J = (0)$, so $B_{k_0 n_0} B_{-k_0 n_0} J = \{0\}$. By the above discussion, $\overline{B_{k_0 n_0} B_{-k_0 n_0}} = J$, and thus $J = (0)$, contradiction. On the other hand, $\{0\} \neq B_{-k_0 n_0} A_{n_0} \subset B_{-(k_0-1)n_0}$. This latter property implies that the assumption that $k_0 > 1$ is the smallest natural number with $B_{-k_0 n_0} \neq \{0\}$ is false, so $k_0 = 1$. Therefore $B_{-n_0} \neq \{0\}$. An induction argument shows that $B_{-kn_0} \neq \{0\}$ for every $k \in \mathbf{N}$. By the first part of the proof, we have that $B_{-n} = A_{-n}$ for every $n \in sp(\alpha)$, $n \neq 0$. Since $A^\alpha \subseteqq B$, it follows that $B = A$ and the proof is complete.

ii) $\Rightarrow$ i) Assume that $A^\alpha([0,\infty))$ is a maximal norm closed subalgebra of A. Let $n_0 \in sp(\alpha)$, $n_0 > 0$ be the least positive element of $sp(\alpha)$. Denote $J_{n_0} = \overline{A_{n_0} A_{-n_0}}$. We will prove first that J_{n_0} is a simple C*-algebra. Let $I \subset J_{n_0}$ be a non zero, two sided ideal. Consider the folowing subspace of A :

$$\mathcal{M} = \overline{lin\,\{A_{-n} I | n \in sp(\alpha), n > 0\} + A^\alpha[0,\infty)}$$

Then, $\mathcal{M}$ is a norm closed, α-invariant subspace of A. Clearly, the set $B = \{a \in A | am \in \mathcal{M}, m \in \mathcal{M}\}$ is a norm closed subalgebra of A and $A^\alpha([0,\infty)) \subset B$. Since $\{0\} \neq I \subset J_{n_0}$, it follows that $J_{n_0} I = I \neq \{0\}$ and therefore there are $a \in A_{-n_0}$ and $i_0 \in I$ such that $b = ai_0 \neq 0$. Then, since n_0 is the least positive element of $sp(\alpha)$, it is clear that $bm \in \mathcal{M}$ for every $m \in \mathcal{M}$. Hence $b \in B$. Obviously, $b \notin A^\alpha([0,\infty))$ and thus B is a norm closed subalgebra of A such that $A^\alpha([0,\infty)) \subsetneqq B$. Since $A^\alpha([0,\infty))$ is by assumption a maximal norm closed subalgebra of A, it follows that $B = A$. Therefore, in particular, $A_{-n_0} A^\alpha \subset \mathcal{M}$.

This means that $\overline{A_{-n_0}A^\alpha} \subset \overline{A_{-n_0}I}$ which implies that $J_{n_0} \subset I$ and so $I = J_{n_0}$. Therefore J_{n_0} is a simple C*-algebra as claimed and

$$\mathcal{M} = \overline{lin\,\{A_{-n}J_{n_0} | n \in sp(\alpha), n > 0\} + A^\alpha[0,\infty)}$$

Let now $n_1 \in sp(\alpha)$, $n_1 \geqq 1$ be an arbitrary positive element of $sp(\alpha)$, so $n_1 \geq n_0$. Since $B = A$, we have $A_{-n_1}\mathcal{M} \subset \mathcal{M}$, so, in particular $A_{-n_1}A^\alpha \subset \overline{A_{-n_1}J_{n_0}}$. By multiplying the latter equality to the right by A_{n_1} we get $J_{n_1} = \overline{A_{n_1}A_{-n_1}A^\alpha} \subset J_{n_1}J_{n_0} \subset J_{n_0}$. Since J_{n_0} is simple, it follows that $J_{n_1} = J_{n_0}$ and therefore the condition (S) holds. $\blacksquare$

In the particular case when A is the crossed product $A = C \ltimes_\beta \mathbf{Z}$, where C is simple, a similar result was obtained in [13].

Next, we will discuss in more detail the structure of a C*-dynamical system $(A, \mathbf{T}, \alpha)$ which satisfies the condition ii1) in Proposition 12. Let $(A, \mathbf{T}, \alpha)$ be a C*-dynamical system with $sp(\alpha) = \{-n_0, 0, n_0\}$ for some $n_0 \in \mathbf{N} \subset \mathbf{Z}$ such that the ideal $I = \overline{A_{n_0}A_{-n_0}} \subset A^\alpha$ is a simple C*-subalgebra of A^α so that the condition ii1) of Proposition 12 is satisfied. Let $J = \overline{A_{-n_0}A_{n_0}}$. Then, A_{n_0} is an $I - J$ imprimitivity bimodule in the sense of Rieffel [16] and therefore, J is strongly Morita equivalent with I. Hence, J is also a simple C*-subalgebra of A^α. Moreover, since $sp(\alpha)$ does not contain $2n_0$ it follows that $IJ = \{0\}$.

Consider the C*-subalgebra $B \subset A$, $B = \overline{(I+J)A(I+J)}$. Then, obviously, B is an $\alpha-$invariant hereditary C*-subalgebra of A, $B_{n_0} = A_{n_0}$ and $B^\alpha = I + J$. It is easy to show that $\widetilde{\Gamma}(\alpha) = \{0\}$. The following Proposition describes the C*-dynamical system in this case:

Proposition 14 *If $(A, \mathbf{T}, \alpha)$ and B are as above, then B is a simple C*-algebra.*

Proof. We will prove first that B is $\alpha-$simple, that is if $\{0\} \neq K \subset B$ is a norm-closed $\alpha-$invariant ideal of B, then $K = B$. Let $\{0\} \neq K \subset B$ be such an ideal. Then $K^\alpha \subset B^\alpha = I + J$ is a two-sided ideal of $I + J$. Since $K \neq \{0\}$, it follows that $K^\alpha \neq \{0\}$. Therefore, either $K^\alpha I \neq \{0\}$, or $K^\alpha J \neq \{0\}$ or both $K^\alpha I \neq \{0\}$ and $K^\alpha J \neq \{0\}$. In the latter case, since both I and J are simple C*-algebras, it follows that $K^\alpha = I + J$. In this case, by applying Remark 7 i) it follows that $A_{n_0} = A_{n_0}J \subset K$ and thus, $K = B$. Next, we will show that the situation $K^\alpha I \neq \{0\}$ and $K^\alpha J = \{0\}$ (respectively $K^\alpha J \neq \{0\}$ and $K^\alpha I = \{0\}$) cannot occur. If $K^\alpha I \neq \{0\}$ and $K^\alpha J = \{0\}$ it follows that $K_{n_0} = \{0\} = K_{-n_0}$. Indeed, if $x \in K_{n_0}$, then $x^*x \in J \cap K^\alpha = \{0\}$, so $x = 0$. It then follows that $sp(\alpha|_K) = \{0\}$, hence $K = K^\alpha = I$. But I is not an ideal of B since $B_{-n_0} = B_{-n_0}I \subsetneqq I$. Therefore, B is $\alpha-$simple. We prove next that B is a simple C*-algebra. Before starting the proof of the claim, notice that since $sp(\alpha)$ is, in particular, a compact subset of $\widehat{\mathbf{T}} = \mathbf{Z}$, by [[12], Theorem 8.1.12.] the action α is uniformly continuous and therefore, the dual and the second dual actions α^* and α^{**} are uniformly continuous as well. Therefore, by [[12], Corollary 8.5.3.], $\alpha^{**}|_{\mathcal{Z}}$ is trivial, where $\mathcal{Z}$ is the center of the second dual B^{**} of B. Let now $\{0\} \neq K \subset B$ be a norm-closed two-sided ideal of B and $e \in B^{**}$ be the corresponding central projection. Since $\alpha^{**}|_{\mathcal{Z}}$ is trivial, it follows that

$\alpha^{**}(e) = e$ and thus K is $\alpha-$invariant. Since B is $\alpha-$simple, it follows that $K = B$ and the proof is complete. ■

Example 15 *Let B be a simple C*-algebra and $A = M_2(B)$, the algebra of 2x2 matrices with entries in B and $\alpha_z\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & zb \\ \overline{z}c & d \end{bmatrix}$, $z \in \mathbf{T}$. Then, the system $(A, \mathbf{T}, \alpha)$ satisfies the hypotheses of the above Proposition.*

Proof. It is easy to show that α is an action of $\mathbf{T}$ on A with fixed point algebra $A^\alpha = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $A_{-1} = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ and $A_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if $n \notin \{-1, 0, 1\}$. Then, if $I = \overline{A_1 A_{-1}} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $a \in B$ and $J = \overline{A_{-1} A_1} = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$, $d \in B$, we have $IJ = \{0\}$ and the condition (S) holds. ■

Example 16 *More generally, let I and J be two strongly Morita equivalent simple C*-algebras. Then, by [[5], Theorem 1.1], there is a C*-algebra B such that I and J are isomorphic with complementary full corners of B. The elements of the algebra B are 2 x 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \in I$, $d \in J$, and, if X is the imprimitivity bimodule which establishes the Morita equivalence, $b \in X$ and $c \in \widetilde{X}$ where $\widetilde{X}$ is the dual of X in the sense of Rieffel [[16], Section 6]. Then if we define $\alpha_z\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & zb \\ \overline{z}c & d \end{bmatrix}$, $z \in \mathbf{T}$, the action α satisfies the hypotheses of the above proposition with $sp(\alpha) = \{-1, 0, 1\}$.*

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